Pixel geometry for a tilted, space borne interferometric radiometer

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ACRONYMS

ESA	European Space Agency
FOV	Field of View
MIRAS	Microware Imaging Radiometer with Aperture Syntheris
SAG	Science Advisory Group
SMOS	Soil Moisture and Ocean Salinity Mission

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1 INTRODUCTION

ESA has decided to conduct a Phase A study on the SMOS proposal, which was submitted in November 1998 in answer to a call for opportunity missions in the frame of the ESA Earth Explorer program.
SMOS aims at measuring surface soil moisture over land surfaces, and sea surface salinity over the oceans. The measuring principle is L-band (1420 MHz) radiometry; the measuring technique is 2D interferometry.
In the course of preparing the mission, requirements concerning the accuracy of retrieved parameters, spatial resolution and revisit time have to be agreed upon, and stipulated to the Phase A contractor.
The present document addresses spatial resolution, which is of considerable importance over land surfaces. This issue was discussed at length by the Science Advisory Group (SAG for the SMOS mission). To this end, two technical notes were prepared for relevant SAG meetings in May and September 2000.
However, these brief notes do no go into the detailed calculation of spatial resolution parameters. Nonetheless, expliciting this calculation may turn out to be useful, in order to make sure that every agent during the Phase A (including the Phase A main contractor) uses the same, or at least consistent, definitions and approach.
This is the purpose of the present work. Section 2 delineates the problem; section 3 provides explicitly the method used in order to determine analytically the SMOS pixel features, assuming an elliptical shape. Section 4 briefly discusses the results.
NOTE: parts of the following sections have been borrowed, with some modifications, from the notes written for the SMOS SAG.

2. DELINEATING THE PROBLEM

2.1 GENERAL REMARKS

Unlike a radar where spatial resolution may be determined by time sampling or Doppler characteristics, in a radiometer the spatial resolution at ground level is entirely dictated by the angular resolution of the (copolar) antenna pattern.
A useful way of characterizing this angular resolution is to consider the 3 dB solid angle, i.e. the solid angle within which the directional (power) gain is larger than half the gain on axis. Accordingly, the pixel will be defined as the intersection of the cone including this solid angle with Earth's surface.
It is however good to remember that the gain integrated over this solid angle amounts, as an order of magnitude, to about two thirds of the total gain ; most of the remaining fraction lies in the wider part of the antenna main beam (since the main beam efficiency is assumed to be high)

2.2 THE SMOS CASE

In the case of a 2D interferometric radiometer, several complications arise.

Going into the details of interferometry is beyond the purpose of this note. Very shortly, the interferometric device combines signals collected by a number of elementary radiating antennas, in order to build correlation products; next, these products can be used to reconstruct a map of brightness temperatures over a solid angle, through a transform which, in an ideal case, would simply reduce to an inverse Fourier transform.

The SMOS instrument follows developments undertaken by ESA within the frame of the MIRAS project. The antenna is a planar, Y shaped structure; each arm is about 4.5 m long and include circa 25 adjacent radiating elements.

In this context, two potential factors on the spatial resolution deserve to be mentioned:

As the baselines available for image reconstruction are shaped as a star rather than a circle (due to the Y shaped antenna), the "size" of the reconstructed antenna is not isotropic. It turns out that while the consequences of this factor may be significant for the secondary lobes, they are negligible as far as the main beam is concerned (E. Anterrieu, private communication).

A further anisotropy in angular resolution may come from the fringe washing function due to the finite frequency bandwidth (see A. Camps, Ph.D. Thesis). The magnitude of the resulting effects seems to be of the order of a few per cent.

On the whole, however, the geometry of the SMOS pixel, as defined above, results essentially from the viewing geometry. Consequently, the map of pixels sizes across the instantaneous, 2 D field of view (FOV) is mostly determined by the antenna size and apodization function (which together determine the angular resolution on axis), the tilting angle and the flight altitude.

2.3 HORIZONTAL ANTENNA PLANE

The antenna pattern to be considered is the "reconstructed" antenna pattern. On the axis (i.e. for a nadir view), the half power beam width ε_0 will be isotropic:

$$\varepsilon_0 = k \lambda / (2 L)$$

Where λ is the wavelength (21 cm), L is the arm length, and the factor k takes values ranging from 1 to 2, depending on the apodization window. The angular 3 dB beam width for SMOS on axis is expected to be close to 2°.

The instantaneous field of view is **bidimensional**. As a consequence the angular resolution is not isotropic and its characteristic sizes vary all over the field of view; so do the spatial resolutions.

The basic phenomenon is illustrated by Figure 1, which considers a nadir pointed antenna (horizontal antenna plane). Looking at nadir, the pixel is isotropic; the spatial resolution Δs is given by:

$$\Delta s = H \varepsilon_0$$

Where H is the satellite altitude

Looking at a point T on the surface, away from nadir, the "**transverse**" angular half power beam width, **perpendicular** to the incidence plane, is unchanged with respect to nadir; the corresponding spatial resolution Δq becomes:

$$\Delta q = ST \varepsilon_0$$

where ST is the length of the line between the satellite S and the target point T. **If the Earth were a plane**, one would have ST = H / cos(θ), where θ is the look angle (with respect to the vertical through the satellite).

Within the incidence plane, the "**radial**" beam width ε_p is larger, because the relevant size of the antenna is smaller: $\varepsilon_p = \varepsilon_0 / \cos(\theta_c)$, where θ_c is the look angle with respect to the antenna axis. On Figure 1, since the antenna plane is horizontal, $\theta_c = \theta$.

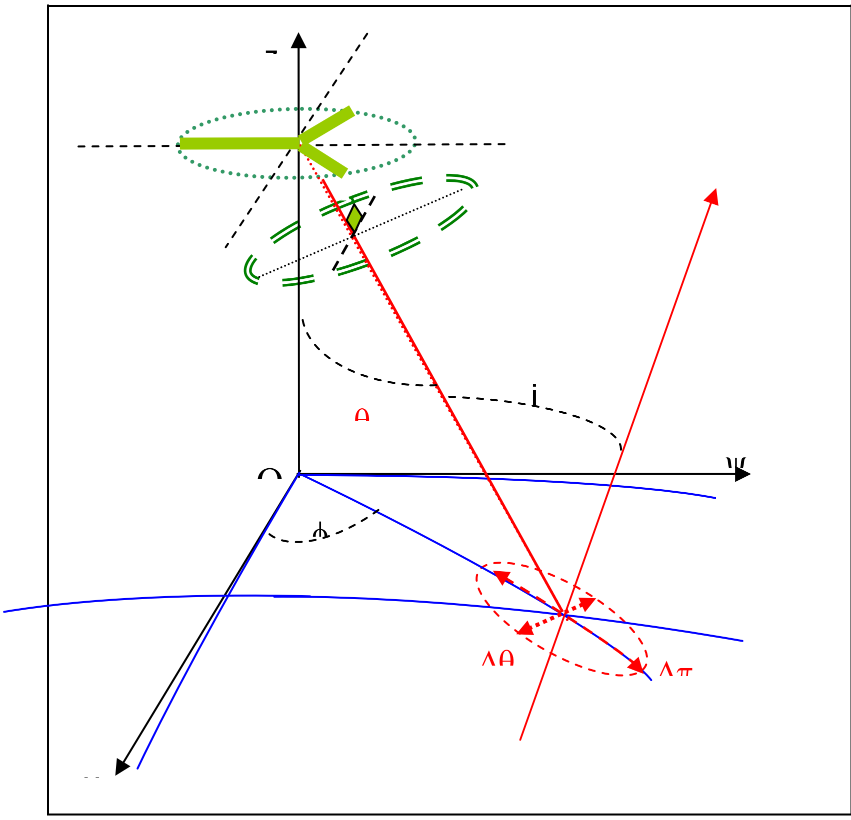

Figure 1 :Geometry

For a **horizontal** antenna plane, the pixel (limited by 3dB directional gain) is approximately an ellipse, with major axis within the elevation plane SOT.

The double-dashed ellipse illustrates the antenna equivalent area when projected on a plane perpendicular to the line of sight

In addition, the radial spatial resolution Δp is multiplied by a factor 1 / cos(i), where i is the incidence angle, and therefore:

$$\Delta\pi= \varepsilon_0\ \Sigma T / (\chi o\sigma(\theta)\ \chi o\sigma(\iota)).$$

If the surface of the Earth were a plane, one would have (with i = θ) :

$$\Delta\theta = \varepsilon_0\ H / \chi o\sigma(\theta)\ ; \qquad \Delta\pi = \varepsilon_0\ H / \chi o\sigma^3(\theta)$$

Although of course the sphericity of the Earth must be taken into account, this helps to appreciate that the actual pixel will be significantly elongated when looking away from the vertical.

It cannot be very wrong to assimilate the surface limiting the half power solid angle to a cone with an elliptical cross section. Even then, the actual pixel on Earth surface will be the intersection of such an elliptical cone with a sphere, which is a complicated curve. Again, the result cannot be very different from an ellipse (this has been verified), and we shall assume here this to be true.

3 TILTED ANTENNA PLANE

3.1 NOTATIONS

When the antenna axis is tilted with respect to the vertical, both look angles θ_c and θ become different. The coordinate system is given by Figure 2.

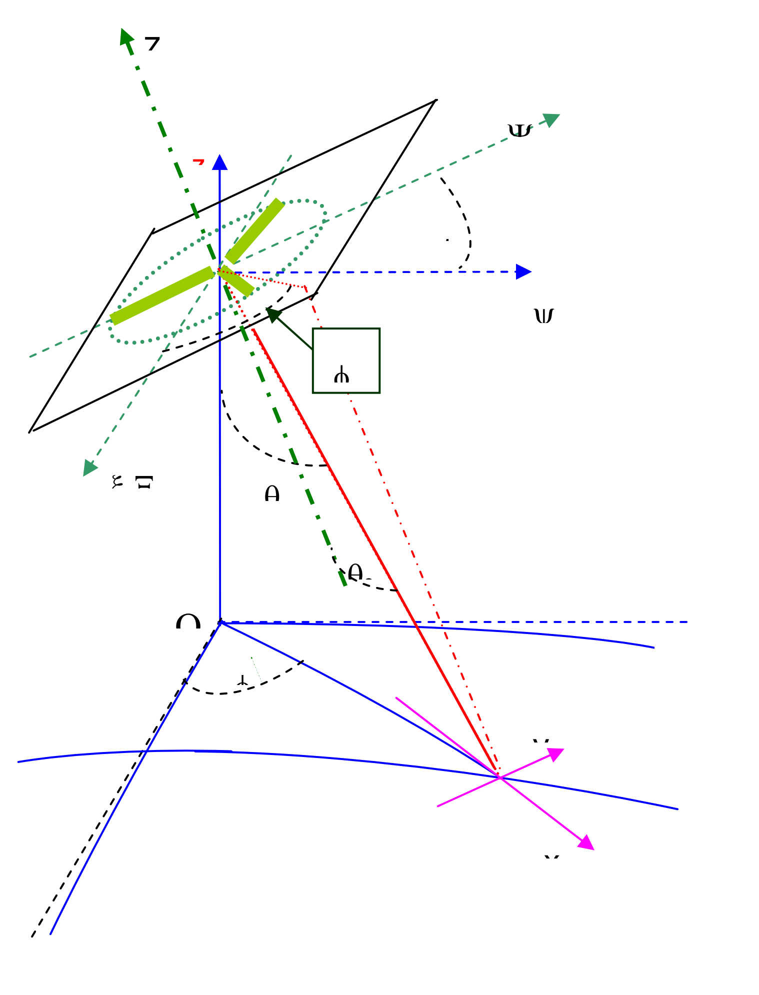

Figure 2 geometry of earth's observation

Geometry of the Earth radiometric observation using an interferometric antenna.

The figure shows the three reference frames used in the calculation, as well as relevant angles.

The **main** reference frame Sxyz is centered on the satellite (Sz axis upwards ; (Sz,Sy) is the orbital plane). H is the flight altitude (H=SO, with O subsatellite point).

The antenna plane is tilted with respect to the horizontal (Sz, Sy) plane by the **tilt angle** t ; the antenna axis lies in the (Sz, Sy) plane.

The **antenna** frame of reference is SXYZ; SZ is along the antenna axis (oriented upwards); SX is identical to Sx.

Let us consider a point P, the distance of which to S is unity. Then, in both reference frames:

$$X = \sin(\theta_a)\cos(\phi_a) \qquad x = X$$

$$(1) \qquad Y = \sin(\theta_a)\sin(\phi_a) \qquad y = Y\cos(t) - Z\sin(t)$$

$$Z = \cos(\theta_a) \qquad z = Y\sin(t) + Z\cos(t)$$

where θ_a and ϕ_a are angular coordinates (elevation and azimuth) in the antenna reference frame (note that, due to the Z axis pointing upwards, $\theta_a = \pi - \theta_c$; $\phi_a = \phi_c$).

Angular coordinates in the main reference system are also defined:

(2) $\tan(\theta) = - m/z$; $\cos(\phi) = x / m$; $\sin(\phi) = y / m$

where m=sqrt $(x^2 + y^2)$

Line SP intersects the terrestrial surface at target point T. The vertical directions at point O (subsatellite point) and T deviate from each other by a small angle ε, because the terrestrial surface is spherical. Standard geometry shows that ε is given by:

(3) $\varepsilon = - \theta + \mathrm{Arcsin}\left((1 + H/R) \sin(\theta) \right)$ where R is the Earth radius (assumed spherical)

The antenna to target distance along the line of sight r = ST is:

(4) $$r = \sqrt{R^2 + (H+R)^2 - 2R(H+R)\cos(\varepsilon)}$$

3.2 NUMERICAL SIMULATION

When the antenna axis is tilted with respect to the vertical, the intersections with Earth's surface of the incidence plane and the plane defined by the line of sight and the antenna axis become different .The result is that the major axis of the ellipse becomes oriented somewhere between the directions of those intersections, and that the ellipse tends to become "shorter", (less elongated), and "thicker".

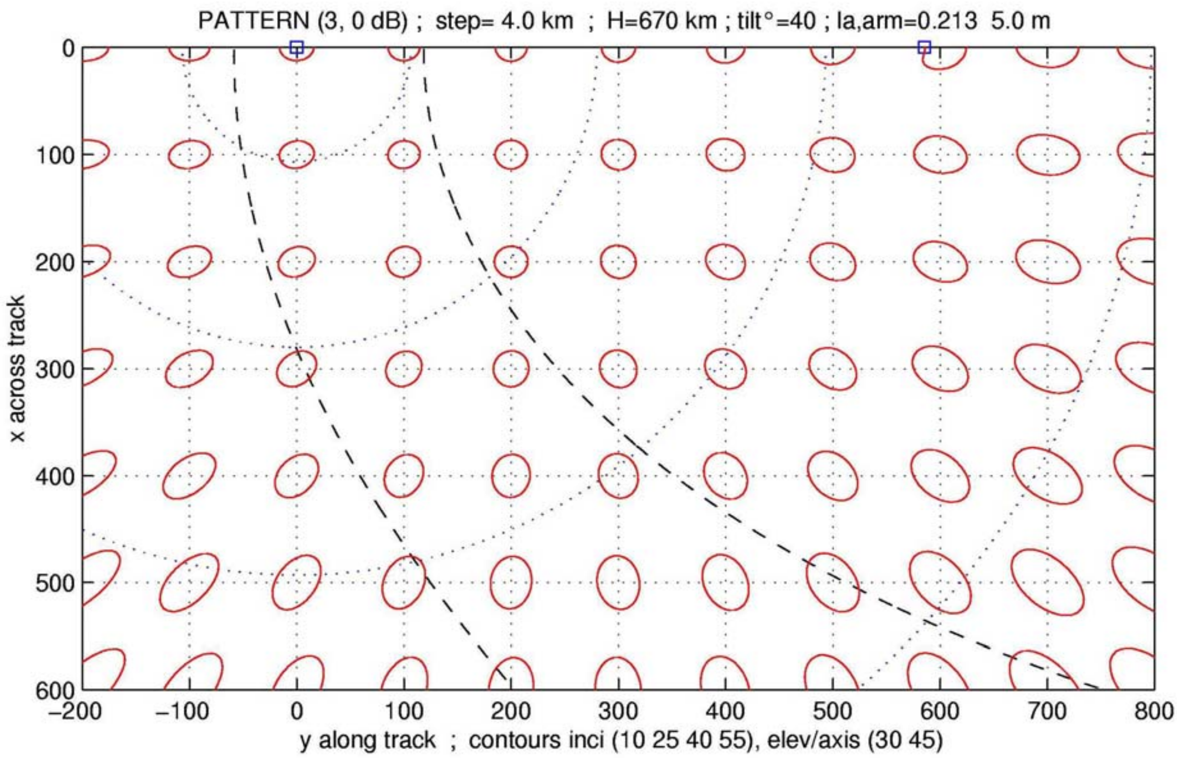

Figure 3: pixel sizes of the r.h.s. of the FOV. The ground track is the horizontal axis at the top of the graph ; the tilting angle is 40°.

This is illustrated by Figure 3, which was obtained through a direct simulation, i.e. a simulation in which the curve where an elliptical cone intersects the Earth is found numerically, then fitted to an ellipse.
Due to the high tilting angle, the pixel is less elongated in the zone (around the centre of the graph) where incidence angle and angle with respect to the antenna axis effects play against each other.
The following paragraph describes how the characteristics of such pixels, approximated by ellipses, are computed analytically.

3.3 FROM ANGULAR COORDINATES TO THE LOCAL REFERENCE FRAME

Let $(\vec{x}_l, \vec{y}_l, \vec{z}_l)$ be the local reference frame at point T (where $\vec{x}_l$ is tangent to the great circle passing through O and T, and $\vec{z}_l$ is the local vertical direction at point T). The transformation of the coordinates of any vector from the $(\vec{X}, \vec{Y}, \vec{Z})$ to the $(\vec{x}_l, \vec{y}_l, \vec{z}_l)$ reference frames may be expressed as a product of the 3 following rotations:

- 1 rotation of angle t around the $\vec{X}$ axis
- 1 rotation of angle ϕ around the $\vec{z}$ axis
- 1 rotation of angle ε corresponding to the change of vertical directions between points S and T

This may be expressed as:

(5) $$\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \begin{bmatrix} \cos\varepsilon & 0 & -\sin\varepsilon \\ 0 & 1 & 0 \\ \sin\varepsilon & 0 & \cos\varepsilon \end{bmatrix} . \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} . \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

This yields:

(6) $$\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = [K] . \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

with

(7) $$[K] = \begin{bmatrix} c(\varepsilon)c(\phi) & c(\varepsilon)s(\phi)c(t) - s(\varepsilon)s(t) & -c(\varepsilon)s(\phi)s(t) - s(\varepsilon)c(t) \\ -s(\phi) & c(\phi)c(t) & -c(\phi)s(t) \\ s(\varepsilon)c(\phi) & s(\varepsilon)s(\phi)c(t) + c(\varepsilon)s(t) & -s(\varepsilon)s(\phi)s(t) + c(\varepsilon)c(t) \end{bmatrix}$$

where sin( ) and cos( ) have been shortened to s( ) and c( ).
Coordinates of point T (denoted hereafter with subscript T) may be written (from (1)):

(8) $$\begin{aligned} X_T &= r \sin(\theta_a) \cos(\phi_a) \\ Y_T &= r \sin(\theta_a) \sin(\phi_a) \\ Z_T &= r \cos(\theta_a) \end{aligned}$$

And thus, by differentiating equations (8):

(9) $$\begin{aligned} dX_T &= \sin(\theta_a)\cos(\phi_a)dr + r\cos(\theta_a)\cos(\phi_a)d\theta_a - r\sin(\theta_a)\sin(\phi_a)d\phi_a \\ dY_T &= \sin(\theta_a)\sin(\phi_a)dr + r\cos(\theta_a)\sin(\phi_a)d\theta_a + r\sin(\theta_a)\cos(\phi_a)d\phi_a \\ dZ_T &= \cos(\theta_a)dr - r\sin(\theta_a)d\theta_a \end{aligned}$$

which may be rewritten as :

(10) $$\begin{bmatrix} dX_T \\ dY_T \\ dZ_T \end{bmatrix} = [M]. \begin{bmatrix} d\theta_a \\ d\phi_a \\ dr \end{bmatrix}$$

with

(11) $$[M] = \begin{bmatrix} r\cos(\theta_a)\cos(\phi_a) & -r\sin(\theta_a)\sin(\phi_a) & \sin(\theta_a)\cos(\phi_a) \\ r\cos(\theta_a)\sin(\phi_a) & r\sin(\theta_a)\cos(\phi_a) & \sin(\theta_a)\sin(\phi_a) \\ -r\sin(\theta_a) & 0 & \cos(\theta_a) \end{bmatrix}$$

Differentiating next equation (6) applied to point T and using equation (10):

(12) $$\begin{bmatrix} dx_{lT} \\ dy_{lT} \\ dz_{lT} \end{bmatrix} = [K].[M]. \begin{bmatrix} d\theta_a \\ d\phi_a \\ dr \end{bmatrix}$$

We now assume that the footprint of the narrow, synthesized antenna beam lies within the plane tangent to the terrestrial surface at point T. Therefore the footprint may be defined by stating that it wholly belongs to this plane:

(13) $dz_{lT} = 0$

The third row of equation (12), combined with (13), then gives:

(14) $\alpha\, d\theta_a + \beta\, d\phi_a + \gamma / r\, dr = 0$

Where $[\alpha \quad \beta \quad \gamma]$ is the bottom row of the matrix $[K].[M]$.

Equations (7) and (11) lead to following expressions:

(15)
$$\begin{aligned}
\alpha = {} & s(\varepsilon)\, c(\phi)\, c(\theta_a)\, c(\phi_a) + s(\varepsilon)\, s(\phi)\, c(t)\, c(\theta_a)\, s(\phi_a) \\
& + c(\varepsilon)\, s(t)\, c(\theta_a)\, s(\phi_a) + s(\varepsilon)\, s(\phi)\, s(t)\, s(\theta_a) - c(\varepsilon)\, c(t)\, s(\theta_a) \\
\beta = {} & - s(\varepsilon)\, c(\phi)\, s(\theta_a)\, s(\phi_a) + s(\varepsilon)\, s(\phi)\, c(t)\, s(\theta_a)\, c(\phi_a) \\
& + c(\varepsilon)\, s(t)\, s(\theta_a)\, c(\phi_a) \\
\gamma = {} & s(\varepsilon)\, c(\phi)\, s(\theta_a)\, c(\phi_a) + s(\varepsilon)\, s(\phi)\, c(t)\, s(\theta_a)\, s(\phi_a) \\
& + c(\varepsilon)\, s(t)\, s(\theta_a)\, s(\phi_a) - s(\varepsilon)\, s(\phi)\, s(t)\, c(\theta_a) + c(\varepsilon)\, c(t)\, c(\theta_a)
\end{aligned}$$

Eliminating dr from equations (12) and (14) allows expressing dx_{lT} and dy_{lT} in terms of $d\theta_a$ and $d\phi_a$ alone. This yields:

(16) $$\begin{bmatrix} dx_{lT} \\ dy_{lT} \end{bmatrix} = [L]. \begin{bmatrix} d\theta_a \\ d\phi_a \end{bmatrix}$$

with

(17) $$[L] = r[K_1].\left\{[M_1] - \frac{1}{\gamma}[M_2]\right\}$$ ([L] is a 2x2 matrix)

and

(18) $$[K_1] = \begin{bmatrix} c(\varepsilon)c(\phi) & c(\varepsilon)c(t)s(\phi) - s(\varepsilon)s(t) & -s(t)c(\varepsilon)s(\phi) - c(t)s(\varepsilon) \\ -s(\phi) & c(\phi)c(t) & -c(\phi)s(t) \end{bmatrix}$$

(19) $$[M_1] = \begin{bmatrix} \cos(\theta_a)\cos(\phi_a) & -\sin(\theta_a)\sin(\phi_a) \\ \cos(\theta_a)\sin(\phi_a) & \sin(\theta_a)\cos(\phi_a) \\ -\sin(\theta_a) & 0 \end{bmatrix}$$

(20) $$[M_2] = \begin{bmatrix} \alpha\sin(\theta_a)\cos(\phi_a) & \beta\sin(\theta_a)\cos(\phi_a) \\ \alpha\sin(\theta_a)\sin(\phi_a) & \beta\sin(\theta_a)\sin(\phi_a) \\ \alpha\cos(\theta_a) & \beta\cos(\theta_a) \end{bmatrix}$$

3.4 THE ELLIPTIC PIXEL

What we are interested in is the pixel shape in the plane tangent to the Earth surface at the target point T. From equation (16), the coordinates (δx_l, δy_l) of a point in the local frame at T are related to angular distances ($\delta\theta_a$, $\delta\phi_a$) with respect to the (θ_a, ϕ_a) angular coordinates of T according to:

(21) $$\begin{bmatrix}\delta x_l \\ \delta y_l\end{bmatrix} = [L].\begin{bmatrix}\delta\theta_a \\ \delta\phi_a\end{bmatrix}$$

In the solid angle, the (assumed elliptical) cone limiting the angular resolution is written (in the antenna frame):

(22) $$\begin{bmatrix}\delta\theta_a & \delta\phi_a\end{bmatrix}\begin{bmatrix}1/a_\theta^2 & 0 \\ 0 & 1/a_\phi^2\end{bmatrix}.\begin{bmatrix}\delta\theta_a \\ \delta\phi_a\end{bmatrix} = 1$$

Where a_θ and a_ϕ are the half main axes of the angular ellipse.

The ellipse equation can then be transformed, using (21), into the $(T, \vec{x}_l, \vec{y}_l)$ frame:

(23) $$\begin{bmatrix}\delta x_l & \delta y_l\end{bmatrix}.[W].\begin{bmatrix}\delta x_l \\ \delta y_l\end{bmatrix} = 1$$

with

(24) $$[W] = \left[L^{-1}\right]^t \begin{bmatrix}1/a_\theta^2 & 0 \\ 0 & 1/a_\phi^2\end{bmatrix}.\left[L^{-1}\right]$$

The eigenvectors of the 2 x 2 matrix [W] yield the orientations of the main ellipse axes: let us write the above ellipse equation (23) after the matrix product has been carried out:

(25) $$A\,(\delta x_l)^2 + 2\,B\,(\delta x_l)(\delta y_l) + C\,(\delta y_l)^2 = 1$$

Then the angle δ of the first axis with respect to the Tx_l axis is found by rotating the reference frame by δ and cancelling the cross-product term. The angle δ and the lengths E1, E2 of the main axes are given by:

(26)
$$\delta = \arctan\,[2\,B/(A - C)\,]$$
$$E1 = [\,A\cos^2(\delta) + 2\,B\sin(\delta)\cos(\delta) + C\sin^2(\delta)\,]^{-1/2}$$
$$E2 = [\,A\sin^2(\delta) - 2\,B\sin(\delta)\cos(\delta) + C\cos^2(\delta)\,]^{-1/2}$$

It remains to specify the angular widths α_θ and α_ϕ used in equation (22).
Let us begin with the synthesized antenna beam along the bore sight normal to the antenna plane: this beam is roughly circular (neglecting side lobes), with an angular width w determined by the apodization function.
When the beam is slanted by θ_a with respect to bore sight, the angular width w_t "**transverse**" to the elevation plane (in the antenna frame) remains the same; it is readily seen that the a_ϕ half width, which expresses the variation of angle ϕ, is obtained by dividing 0.5 w by $\sin(\theta_a)$.
The "**radial**" angular width w_r is increased to $w_r = w/\cos(\theta_a)$ since the apparent size of the antenna is reduced by a $\cos(\theta_a)$ factor; the angular half width a_θ is then equal to $0.5\,w_r = 0.5\,w / \cos(\theta_a)$;

4. RESULTS

The calculation described in the above section has been (successfully!) checked against the direct simulation.
As suggested in the annexes, the SMOS Science Advisory Group eventually recommended that the requirements for spatial resolution over land surface, in order to define the part of the field of view to be used for soil moisture retrievals, should be stipulated as follows:

the **geometrical mean** <**s**> of the axes of the 3 dB ellipse (the "pixel size") should not exceed a given limit (probably 50 km) ;

the **axis ratio e** (i. e. the pixel "elongation") should not exceed 1.5

Actually, depending on other mission parameters, it may happen that either will be the most severe. Certainly, one should try to achieve a configuration in which both criteria set similar limits to the FOV. Note that while the <**s**> criterion depends on tilt angle, flight altitude and arm length, the **e** criterion does **not** depend on arm length.
The limits set to the FOV by spatial resolution requirements will obviously and directly restrict the width of the usable FOV, i.e. the swath. But they will, in the general case, have an indirect additional effect, because they decrease the length, **along the satellite track,** within which the same area on the Earth surface can be seen for various incidence angle values. As a consequence, the accuracy on retrieved parameters (which depends on this length) will worsen as the spatial resolution requirements become more severe.
Finally, the basic retrieval scheme consists of combining views of the same area on the surface obtained for various incidence angles as the satellite moves along. Then it becomes clear that the pixel sizes and orientations for all those views are **not the same**. This problem is among the many issues that remained to be dealt with in order to achieve successfully the SMOS objectives.